Applied Mathematics Letters 22 (2009) 401–405

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Superconvergence of functional approximation methods for integral equations

Guangqing Long[a,b,∗], Gnaneshwar Nelakanti[c]

[a] Department of Mathematics, Guangxi Normal College, Nanning 530001, PR China

[b] Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing, 100080, PR China

[c] Department of Mathematics, Indian Institute of Technology Kharagpur, West Bengal-721302, India

ARTICLE INFO

Article history:
Received 12 October 2006
Received in revised form 4 April 2008
Accepted 3 June 2008

Keywords:
Integral equation
Superconvergence
Functional
Collocation
Multi-projection methods

ABSTRACT

In this work, a functional approximation method for calculating the linear functional of the solution of second-kind Fredholm integral equations is developed. When the method is applied to the collocation method or to the multi-projection method, it generates approximations which exhibit superconvergence.

1. Introduction

This work discusses the calculation of linear functionals of the solution u of the second-kind integral equation

$$u - \mathcal{K}u = f, \tag{1.1}$$

where $\mathcal{K}$ is a compact linear integral operator from Banach space X to itself, f is a given function. These types of equations cover many important applications. There is a huge and growing literature on numerical methods for solving the Eq. (1.1), and some of these methods exhibit superconvergence (see, for example, [1,2,5,7,11,12]). In [3,4], a theoretical framework was developed for the analysis of convergence for projection methods and superconvergence for iterated projection methods. Recently, projection methods of a new kind, the so-called multi-projection method and its discrete versions, were developed in [6,9], and exhibited global superconvergence results.

In many physical phenomena and applications, we are often not so much interested in the solution u itself, but rather in the calculation of some linear functional of u, such as the inner product (g, u), where g is given, or the point value of $u(s_0)$, with given point s_0 (see, [1,8,10,12]). [10] developed an approximation method associated with Galerkin methods for calculation of upper and lower bounds for linear functionals of the solutions u, and the order of accuracy of prediction is four times the rate of convergence of the best approximation of the spline subspaces for sufficiently smooth kernels.

The purpose of this work is to develop a superconvergent method, which is called a *functional approximation method* (or simply **FAM**), for calculating the linear functional of the solutions of (1.1). When we apply this method to collocation methods and multi-projection methods, we obtain approximations which exhibit superconvergence. We organize this work as follows. In Section 2, we first describe functional approximation methods, and then we apply this method to collocation methods and multi-projection methods to obtain superconvergence of the linear functional of the solution in Section 3. Section 4 is devoted to a presentation of numerical examples.

∗ Corresponding author at: Department of Mathematics, Guangxi Normal College, Nanning 530001, PR China.
E-mail addresses: longgq@gxtc.edu.cn (G. Long), gnanesh@maths.iitkgp.ernet.in (G. Nelakanti).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.06.007

2. Functional approximation methods

In this section, we introduce functional approximation methods for Fredholm integral equations of the second kind. To this end, we let X be a Banach space with its norm $\|\cdot\|$. We consider the equation

$$u - \mathcal{K}u = f, \tag{2.1}$$

where $\mathcal{K} : X \to X$ is a compact linear integral operator defined by $(\mathcal{K}u)(s) = \int_D K(s,t)u(t)\mathrm{d}t,\ s \in D$, and $D \subset R^d$, $d \geq 1$ is a bounded closed domain. For any function $g \in X$, we also consider the equation adjoint to (2.1)

$$v - \mathcal{K}^* v = g, \tag{2.2}$$

where $\mathcal{K}^*$ is a compact integral operator from X to X, which satisfies $(\mathcal{K}^* v)(s) = \int_D K(t,s)v(t)\mathrm{d}t$. We assume that the solutions of (2.1) and (2.2) uniquely exist.

We are interested in calculating the linear functional of the solution u. To do this, we define the linear function $\mathcal{F} : X \to C$ as the bilinear form

$$\mathcal{F}(g) := \langle g, u\rangle = \int_D g(s)u(s)\mathrm{d}s, \tag{2.3}$$

for any $g \in X$. It follows from (2.1) that $u = \mathcal{K}^2 u + \mathcal{K}f + f$, which leads to

$$\mathcal{F}(g) = \left\langle g, \mathcal{K}^2 u\right\rangle + \langle g, (\mathcal{I} + \mathcal{K})f\rangle = \mathcal{H}(g) + \mathcal{E}(g), \tag{2.4}$$

where $\mathcal{H}(g) := \left\langle g, \mathcal{K}^2 u\right\rangle$ and $\mathcal{E}(g) := \langle g, (\mathcal{I} + \mathcal{K})f\rangle$.

In order to calculate (2.3), we develop a superconvergent method by constructing functional approximations for $\mathcal{F}(g)$. In this work we call such a method *the functional approximation method* (FAM). To do this, we let $\mathcal{K}_n$ and $\widetilde{\mathcal{K}}_n$ be approximation operators for $\mathcal{K}$ and $\mathcal{K}^*$ in norm, respectively, and consider the approximation schemes for (2.1) and (2.2), that is, seeking unknown functions $u_n \in X$ and $v_n \in X$ such that

$$u_n - \mathcal{K}_n u_n = f_n, \tag{2.5}$$

and

$$v_n - \widetilde{\mathcal{K}}_n v_n = g_n, \tag{2.6}$$

where f_n and g_n are the approximations of f and g. We assume that the Eqs. (2.5) and (2.6) have unique solutions u_n and v_n, respectively.

Making use of the approximation solutions u_n and v_n, we define the approximation function $\mathcal{F}_n : X \to C$ for $\mathcal{F}$ by

$$\mathcal{F}_n(g) = \mathcal{H}_n(g) + \mathcal{E}(g), \quad \text{for any } g \in X, \tag{2.7}$$

where $\mathcal{H}_n(g) := \left\langle g, \mathcal{K}^2 u_n\right\rangle + \left\langle \mathcal{K}^{*2} v_n, f - (\mathcal{I} - \mathcal{K})u_n\right\rangle$.

In the following theorem, we provide the estimate of the error between $\mathcal{F}_n(g)$ and $\mathcal{F}(g)$.

Theorem 2.1. *Let $\mathcal{K}_n$ and $\tilde{\mathcal{K}}_n$ be the approximation operators for $\mathcal{K}$ and $\mathcal{K}^*$ in norm, and u, v be the unique solutions of (2.1) and (2.2), respectively. Then for any $g \in X$, there exists a constant c independent of n such that*

$$|\mathcal{F}(g) - \mathcal{F}_n(g)| \leq c\|\mathcal{K}^*(v - v_n)\|\,\|\mathcal{K}(u - u_n)\|. \tag{2.8}$$

Proof. It follows from (2.4) and (2.7) that

$$\begin{aligned}
\mathcal{F}(g) - \mathcal{F}_n(g) &= \left\langle g, \mathcal{K}^2 u\right\rangle - \left\langle g, \mathcal{K}^2 u_n\right\rangle - \left\langle \mathcal{K}^{*2} v_n, (\mathcal{I} - \mathcal{K})(u - u_n)\right\rangle \\
&= \left\langle (\mathcal{I} - \mathcal{K}^*)v, \mathcal{K}^2(u - u_n)\right\rangle - \left\langle \mathcal{K}^{*2} v_n, (\mathcal{I} - \mathcal{K})(u - u_n)\right\rangle \\
&= \left\langle \mathcal{K}^* v, (\mathcal{I} - \mathcal{K})\mathcal{K}(u - u_n)\right\rangle - \left\langle \mathcal{K}^* v_n, (\mathcal{I} - \mathcal{K})\mathcal{K}(u - u_n)\right\rangle \\
&= \left\langle \mathcal{K}^*(v - v_n), (\mathcal{I} - \mathcal{K})\mathcal{K}(u - u_n)\right\rangle.
\end{aligned}$$

Hence we obtain $|\mathcal{F}(g) - \mathcal{F}_n(g)| \leq c\,\|\mathcal{K}^*(v - v_n)\|\,\|\mathcal{K}(u - u_n)\|$, where $c = 1 + \|\mathcal{K}\|$, which completes the proof. □

Using Theorem 2.1, we have the following straightforward corollary.

Corollary 2.2. *Assume the conditions of Theorem 2.1 hold, and choose $g = K_s$, where $K_s = K(s,\cdot)$, $s \in D$, is the kernel of integral operator $\mathcal{K}$ with $K(s,s) \neq 0$. Then for any $s \in D$, it holds that*

$$u(s) = \mathcal{F}(K_s) + f(s). \tag{2.9}$$

Let $u_n'(s) = \mathcal{F}_n(K_s) + f(s)$ be the approximation for the solution u at point $s \in D$. Then it holds that $u(s) - u_n'(s) = \mathcal{F}(K_s) - \mathcal{F}_n(K_s)$, and there exists a constant c independent of n such that

$$|u(s) - u'(s)| \leq c\|\mathcal{K}^*(v - v_n)\|\|\mathcal{K}(u - u_n)\|. \tag{2.10}$$

3. Superconvergence orders

In this section, we choose collocation methods and multi-projection methods to give the approximation schemes for (2.1) and (2.2), and investigate the superconvergence order of the corresponding FAM.

We assume that D is divided into N_n simplices $\Delta_n := \{E_{n,1}, \ldots, E_{n,N_n}\}$ such that $D = \cup_{i=1}^{N_n} E_{n,i}$, for any $i \neq j$, $\mathrm{meas}(E_{n,i}, E_{n,j}) = 0$, and

$$h = h_n := \max\{\mathrm{diam}E_{n,i} : i = 1, 2, \ldots, N_n\} \to 0, \quad \text{as } n \to \infty.$$

Let $X_n \subset X$ be the piecewise polynomial space of total degree $r-1$ related to Δ_n, and assume that $\mathcal{P}_n : X \to X_n$ are bounded linear projection operators converging to the identity operator $\mathcal{I}$ pointwise.

3.1. Collocation methods

Let $X = L^\infty(D)$, and $\mathcal{P}_n : X \to X_n$ be a sequence of bounded interpolation projections such that for each $u \in X$, $\lim_{n\to\infty} \|\mathcal{P}_n u - u\|_\infty = 0$. The approximation schemes (2.5) and (2.6) can be written as seeking $u_n \in X_n$ and $v_n \in X_n$ such that

$$u_n - \mathcal{P}_n\mathcal{K}u_n = \mathcal{P}_n f, \quad \text{and} \quad v_n - \mathcal{P}_n\mathcal{K}^* v_n = \mathcal{P}_n g. \tag{3.1}$$

It is known that for both equations of (3.1) there exist unique solutions [1,2].

We next investigate the error estimate of functional approximations $\mathcal{F}_n(g)$ and $\mathcal{F}(g)$ employed with approximation solutions of (3.1), and the estimate of the point value of the solution u. To this end, we quote the following lemma from [1, 12].

Lemma 3.1. *Assume that $(\mathcal{I} - \mathcal{K})^{-1}$ and $(\mathcal{I} - \mathcal{K}^*)^{-1}$ exist on X, with the kernel $K(\cdot,\cdot) \in C^{2r}(D \times D)$. Let $u, v \in C^{2r}(D)$ and $u_n, v_n \in X_n$ be solutions of (2.1), (2.2) and (3.1). Then there exists a constant c such that*

$$\|\mathcal{K}(u - u_n)\|_\infty \le ch^{2r}, \quad \text{and} \quad \|\mathcal{K}^*(v - v_n)\|_\infty \le ch^{2r}. \tag{3.2}$$

By Theorem 2.1, Corollary 2.2 and Lemma 3.1, we have the following estimates.

Theorem 3.2. *Assume that the conditions of Lemma 3.1 hold. Then for any function $g \in X$, it holds that*

$$|\mathcal{F}(g) - \mathcal{F}_n(g)| = \mathcal{O}(h^{4r}).$$

In particular, when $g = K_s$, where $K_s = K(s,\cdot)$, $s \in D$, then for any $s \in D$ it holds that

$$|u(s) - u_n'(s)| = \mathcal{O}(h^{4r}).$$

3.2. Multi-projection methods with Galerkin methods

In this subsection, we apply multi-projection methods to the Galerkin case for (2.5) and Galerkin methods for (2.6) as approximation schemes, and investigate the corresponding FAM.

To do this, we let $X := L^2(D)$ with $\langle\cdot,\cdot\rangle$ denoting the inner product, and $\mathcal{P}_n : X \to X_n$ be orthogonal projection operators. It is known that there exists a constant c independent of n such that $\|\mathcal{P}_n\|_{L^\infty\to L^\infty} \le c$, and for any $\phi \in C^r(D)$, $\|(\mathcal{I} - \mathcal{P}_n)\phi\|_\infty \le ch^r\|\phi^{(r)}\|_\infty$ (see, for example, [1]). We define approximation operators $\mathcal{K}_n^M : X \to X$ by

$$\mathcal{K}_n^M := \mathcal{P}_n\mathcal{K}\mathcal{P}_n + (\mathcal{I} - \mathcal{P}_n)\mathcal{K}\mathcal{P}_n + \mathcal{P}_n\mathcal{K}(\mathcal{I} - \mathcal{P}_n). \tag{3.3}$$

The multi-projection method(MPM) for solving (2.1) is to seek a function $u_n \in X$ such that

$$u_n - \mathcal{K}_n^M u_n = f, \tag{3.4}$$

and the Galerkin method for solving (2.2) is to seek a function $v_n \in X$ such that

$$v_n - \mathcal{P}_n\mathcal{K}^* v_n = \mathcal{P}_n g. \tag{3.5}$$

It is clear that $\mathcal{K}_n^M$ approximates $\mathcal{K}$ in norm, and the inverses $(\mathcal{I} - \mathcal{K}_n^M)^{-1}$ and $(\mathcal{I} - \mathcal{P}_n\mathcal{K}^*)^{-1}$ exist and are uniformly bounded (see [1,2,6]).

Lemma 3.3. *Assume that $(\mathcal{I} - \mathcal{K})^{-1}$ and $(\mathcal{I} - \mathcal{K}^*)^{-1}$ exist on X, with the kernel $K(\cdot,\cdot) \in C^r(D \times D)$ defined by (3.3). Let $u, v \in C^r(D)$ and $u_n, v_n \in X$ be solutions of (2.1), (2.2), (3.4) and (3.5) respectively. Then there exists a positive constant c independent of n such that*

$$\|\mathcal{K}(u - u_n)\|_\infty \le ch^{4r}, \quad \text{and} \quad \|\mathcal{K}^*(v - v_n)\|_\infty \le ch^{2r}. \tag{3.6}$$

Table 1
Numerical results

n	$\|u(s)-u_n'(s)\|^{CC}$	$Order^{CC}$	$\|u(s)-u_n'(s)\|^{MG}$	$Order^{MG}$
2	6.595608×10^{-5}		1.850513×10^{-6}	
4	4.562466×10^{-6}	3.853619	3.045326×10^{-8}	5.925184
8	2.926003×10^{-7}	3.962810	4.821743×10^{-10}	5.980898
16	1.840629×10^{-8}	3.990659	7.569930×10^{-12}	5.993130
32	1.152269×10^{-9}	3.997649	1.292022×10^{-13}	5.872577

Proof. It follows from $u = (\mathcal{I} - \mathcal{K})^{-1} f$ and $u_n = (\mathcal{I} - \mathcal{K}_n^M)^{-1} f$ that

$$u - u_n = (\mathcal{I} - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)u,$$

which with $\mathcal{K}(\mathcal{I} - \mathcal{K}_n^M)^{-1} = \mathcal{K} + \mathcal{K}(\mathcal{I} - \mathcal{K}_n^M)^{-1}\mathcal{K}_n^M$ leads to

$$\mathcal{K}(u - u_n) = \mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)u + \mathcal{K}(\mathcal{I} - \mathcal{K}_n^M)^{-1}\mathcal{P}_n\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)u. \tag{3.7}$$

Note that

$$\begin{aligned}
\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u\|_\infty &\le c \inf_{y\in X_n} \|K(s,t) - y(t)\|_\infty \|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u\|_\infty \\
&\le ch^{2r}\|(\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u)^{(r)}\|_\infty \\
&\le ch^{2r} \inf_{y\in X_n} \left\|\frac{\partial^r K(s,t)}{\partial s^r} - y\right\|_\infty \|u - \mathcal{P}_n u\|_\infty \le ch^{4r}.
\end{aligned} \tag{3.8}$$

This implies that $\|\mathcal{K}(u - u_n)\|_\infty \le ch^{4r}$. Similarly, we have $\|\mathcal{K}^*(v - v_n)\|_\infty \le ch^{2r}$. □

Theorem 3.4. *Assume that the conditions of Lemma 3.3 hold. Then for any function $g \in X$, it holds that*

$$|\mathcal{F}(g) - \mathcal{F}_n(g)| = \mathcal{O}(h^{6r}).$$

In particular, when $g = K_s$, where $K_s = K(s, \cdot)$, $s \in D$, then for any $s \in D$ it holds that

$$|u(s) - u_n'(s)| = \mathcal{O}(h^{6r}).$$

We remark that under the conditions of Lemma 3.3, the estimate of $|\mathcal{F}(g) - \mathcal{F}_n(g)|$ and $|u(s) - u_n'(s)|$ can reach $\mathcal{O}(h^{8r})$, when we choose the projection schemes of (2.5) and (2.6) both to be multi-projection methods.

4. Numerical examples

We consider the Fredholm integral equations of the second kind

$$u(s) - \mathcal{K}u(s) = f(s), \quad s \in [0, 1],$$

where $(\mathcal{K}u)(s) = \int_0^1 K(s,t)u(t)\mathrm{d}t$ with $K(s,t) = (1/2)\mathrm{e}^{st}$ and $f(s) = \mathrm{e}^{-s}\cos s$.

Let X_n be the space of piecewise constant functions ($r = 1$) with respect to the uniform partition $0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$, with $h = \frac{1}{n}$. We choose $g = K_s$, $s = \frac{3}{512}$ and calculate the estimate of $|\mathcal{F}(g) - \mathcal{F}_n(g)|$, which equals $|u(s) - u_n'(s)|$.

For the collocation method, interpolation points on [0, 1] are given by $t_p = \frac{2p-1}{2n}$, $p = 1, 2, \ldots n$. In Table 1, we give the numerical results and use $|u(s) - u_n'(s)|^{CC}$ and $Order^{CC}$ to represent the error estimate and convergence order of the collocation methods, and $|u(s) - u_n'(s)|^{MG}$ and $Order^{MG}$ for the error estimate and convergence order of the multi-projection methods with Galerkin methods, respectively.

From the numerical results, we observe that the rate of convergence agrees with theoretical estimates, which are $Order^{CC} = 4$ and $Order^{MG} = 6$, respectively.

Acknowledgements

The first author was supported in part by Guangxi Provincial Natural Science Foundation of China under grant 0728044, and the Foundation of the Education Department of Guangxi Province. The second author was supported in part by the Foundation of the Doctoral Program of National Higher Education of China under grant 20030558008, and Sun Yat-sen University under a grant of the postdoctoral program.

References

[1] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, UK, 1997.
[2] M. Chen, Z. Chen, G. Chen, Approximate Solutions of Operator Equations, World Scientific Publishing Co., Singapore, 1997.
[3] Z. Chen, Y. Xu, J. Zhao, The discrete Petrov–Galerkin method for weakly singular integral equations, J. Integral Equations Appl. 11 (1999) 1–35.
[4] Z. Chen, Y. Xu, The Petrov–Galerkin and iterated Petrov–Galerkin method for second kind integral equations, SIAM J. Numer. Anal. 35 (1998) 406–434.
[5] Z. Chen, C.A. Micchelli, Y. Xu, Fast collocation methods for second kind integral equations, SIAM J. Numer. Anal. 40 (2002) 344–375.
[6] Z. Chen, G. Long, G. Nelakanti, The discrete multi-projection method for Fredholm integral equations for the second kind, J. Integral Equations Appl. 19 (2007) 143–162.
[7] Z. Chen, G. Long, G. Nelakanti, Richardson extrapolation of iterated discrete projection methods for eigenvalue approximation, J. Comput. Appl. Math. (in press).
[8] I.G. Graham, G. Chandler, High-order methods for linear functionals of solutions the second kind integral equations, SIAM J. Numer. Anal. 25 (1988) 1118–1137.
[9] N Gnaneshwar, Spectral approximation for integral operators, Ph.D. Thesis, Indian Institute of Technology, Bombay, India, 2003.
[10] F.-K. Hebeker, J. Mika, D.C. Pack, Application of the superconvergence properties of the Galerkin approximation to calculation of upper and lower bounds for linear functionals of solutions of integral equations, IMA J. Appl. Math. 38 (1987) 61–70.
[11] G. Long, G. Nelakanti, Iteration methods for the Fredholm integral equations of second kind, Comput. Math. Appl. 53 (2007) 886–894.
[12] I.H. Sloan, Superconvergence, in: M. Golberg (Ed.), Numerical Solution of Integral Equations, Plenum, New York, 1990, pp. 35–70.